Mathematic Foundation

Seminar 2: Introduction to differentiation - Solution

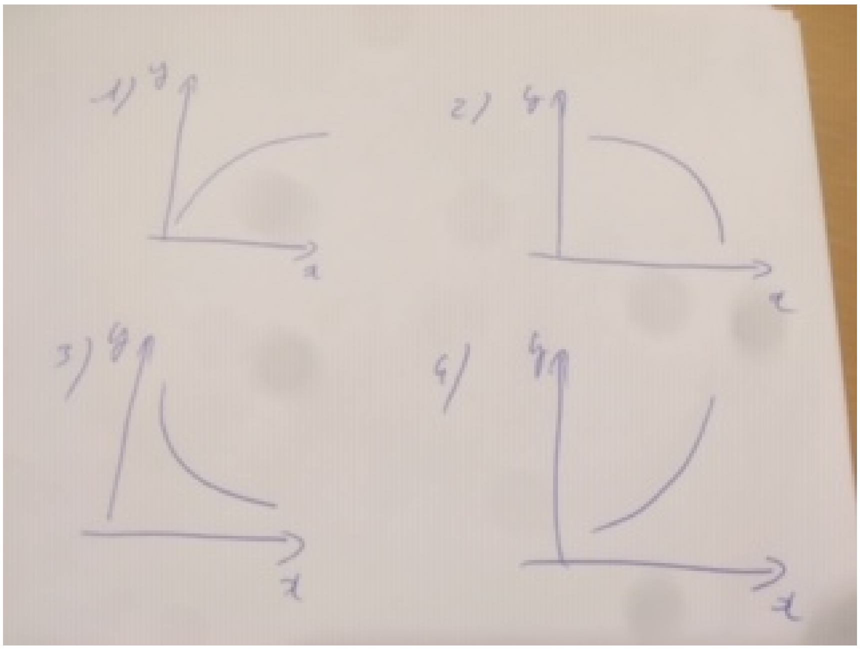
Stockholm School of Economics in Riga, Februrary 2022

Exercise 1

1. $f(x) = x^{1/3}$
   (a) $f'(x) = \frac{1}{3}x^{-2/3}$
   (b) $f''(x) = -\frac{2}{9}x^{-5/3}$
   (c) $f'''(x) = \frac{10}{27}x^{-8/3}$
2. $f(x) = x^4 - 3x^3 + 2x^2 - 8x + 4$
   (a) $f'(x) = 4x^3 - 9x^2 + 4x - 8$
   (b) $f''(x) = 12x^2 - 18x + 4$
   (c) $f'''(x) = 24x - 18$
3. $f(x) = \sqrt{x} - \frac{1}{x}$
   (a) $f'(x) = \frac{1}{2}x^{-1/2} + x^{-2}$
   (b) $f''(x) = -\frac{1}{4}x^{-3/2} - 2x^{-3}$
   (c) $f'''(x) = \frac{3}{8}x^{-5/2} + 6x^{-4}$
4. $f(x) = -3x^2 - 5x + 4$
   (a) $f'(x) = -6x - 5$
   (b) $f''(x) = -6$
   (c) $f'''(x) = 0$

Exercise 2

1. $f' > 0$, $f'' < 0$
2. $f' < 0$, $f'' < 0$
3. $f' < 0$, $f'' > 0$
4. $f' > 0$, $f'' > 0$



Exercise 3

1. $D'(t) = 12t^{1/3}$
2. $D'(8) = 24$
3. $D''(t) = 4t^{-2/3}$
4. $D''(8) = 1$

Exercise 4

1. Stationary point $\Leftrightarrow f'(x) = 0$. To find them, just compute the first derivative and equalize it to 0.
2. Several ways:
   (a) check the value of $f(x)$ just on the right and on the left of the stationary point.
   (b) check the sign of $f'(x)$ just on the right and on the left of the stationary point.
   (c) **use the second derivative test**.

Exercise 5

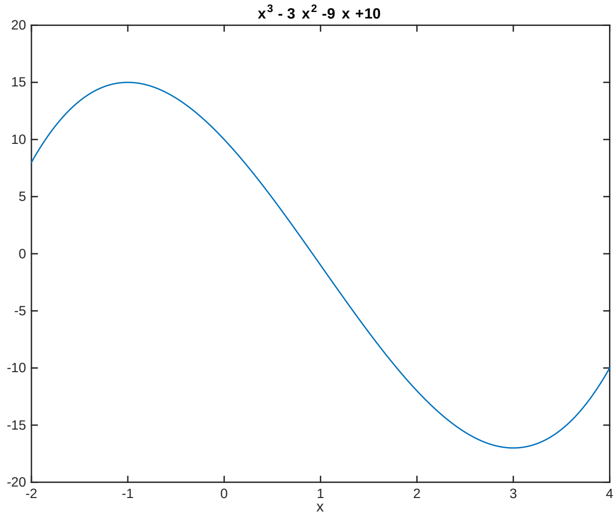
1. The stationary points are the points for which $f'(x) = 0$, hence: $f'(x) = 3x^2 - 6x - 9 = 0$.To find the roots, we can use the quadratic formula. The discriminant is $\Delta = 36 - 4 \times (-27) = 144$. The roots are $\frac{6+12}{6} = 3$ and $\frac{6-12}{6} = -1$. So we have two stationary points $x = -1$ and $x = 3$.

   How can we know if they are minimum, maximum or inflection points? The easiest method is to compute $f'(x)$ for an arbitrary value lower than -1, for another value between -1 and 3, and a third value greater than 3.

   Let's compute $f'(x)$ for $x = -2$, $x = 0$ and $x = 4$:

   - $f'(-2) = 15$, hence $f'(-2) > 0$. Hence, f is increasing for $x < -1$.
   - $f'(0) = -9$, hence f is decreasing for $x \in (-1, 3)$.
   - $f'(4) = 15$, hence f is increasing for $x > 3$.

2. $f(0) = 10$, $f(-1) = 15$ and $f(3) = -17$. We can use all this information to sketch the graph of the function:



Exercise 6

Let's proceed step by step.

- As the plane is on the ground at $x = 0$, we know that $f(0) = 0$. It implies that $d = 0$.

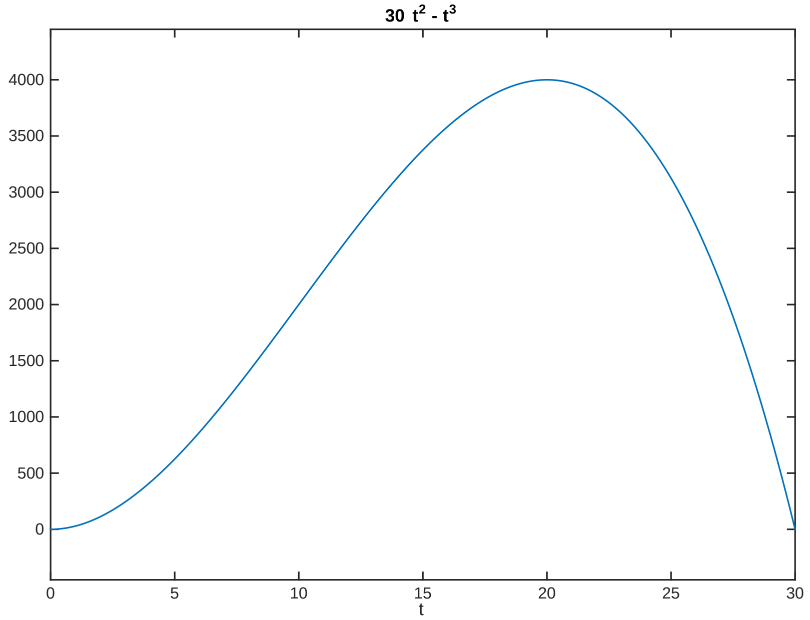
- The plane is horizontal at $x = 0$. It means that $f'(0) = 0$. The derivative of f is $f'(x) = 3ax^2 + 2bx + c$. Hence, $c = 0$, and the polynomial we are looking for is of the form: $f(x) = ax^3 + bx^2$.

- The third statement tells us that $f(100) = 5$, and $f'(100) = 0$. We thus have a system of two equations with two unknowns:

- $\begin{cases} 100^3a + 100^2b = 5 \\ 3a100^2 + 2b100 = 0 \end{cases}$. Solving this system, you should find that $a = -0,00001$ and $b = 0,0015$.

Exercise 7

1. The derivative of f represent the speed of the spread of the disease.
2. $f'(t) = 60t - 3t^2$. Hence, $f'(10) = 600 - 3 \times 10^2 = 300$. At the 10^{th} day, the speed of the spread of propagation is of 300 persons per day.
3. $f'(t) = 0 \Leftrightarrow t = 0$ or $t = 20$. Let's take a t between 0 and 20, say, $t = 1$. $f'(1) = 57$, so $f(t)$ is increasing between 0 and 20. After 20, $f(t)$ is decreasing (you can see that by computing $f'(t)$ for any $x > 20$.
4. As f is increasing up to $x = 20$ and then decreasing, the maximum number of ill persons is reached for $t = 20$. A this date, there are $f(20) = 4000$ ill persons.
5. Here, the question is not whether the function is increasing or not, but what is the *speed* of the increasing. To find the maximum speed of the spread, we need to find the maximum of $f'(t)$. To do so, we can compute second derivative. Here, $f''(t) = -6t+60$. Proceeding as usual, you should find that the maximum of $f'(t)$ is reached for $t = 10$. It means that the disease is spreading at the highest pace during the 10^{th} day.



You can see on the graph that f is increasing more and more up to $x = 10$, then it keeps increasing but the pace is getting slower and slower.