SSE Riga - Maths Foundation

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February 12, 2022

Session 2 : Introduction to optimization

- Last week :
  - What is the rate of change at a point x_0 ?
    $\Rightarrow$ **derivative**

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- Today : how to find the min/max of a function ?
- In economics :
  - What production will maximize profit ?
  - Optimal rate of extraction for an oil company ?
  - Tax rate maximizing tax revenue ?
  - ...

Definition

- At **local maximum** $\hat{x}$,

$$f(\hat{x}) \geq f(x) \ , \ \ \hat{x} - \epsilon \leq x \leq \hat{x} + \epsilon$$

  for x in an interval around $\hat{x}$.
- At a **global maximum** x^*,

$$f(x^*) \geq f(x) \text{ for all } x.$$

Necessary first-order condition

- Suppose that a function f is differentiable in an interval I and that c is an interior point of I
- c is a **stationary point** if $f'(c) = 0$
- For $x = c$ to be a maximum or minimum point for f in I, a necessary condition is that it is a **stationary point** for f

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- A stationary point may be a an extreme value **OR** an inflection point

First-derivative test for minimum/maximum

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  $\Rightarrow$ YES !

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- Often, this is not convenient to implement with more complicated functions. Is there an alternative?
  $\Rightarrow$ YES!
- We first need to introduce higher-order derivatives.

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- Can we go further ?

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- Can we go further ?
- The derivative of the derivative is the **second derivative**
- Differentiating again gives the **third derivative**, etc.
- If $f'(x_0)$, $f''(x_0)$, ..., $f^{(n)}(x_0)$ all exist, then we say that f is n times differentiable at x_0

Higher-order derivatives

- What does the second derivative mean ?
- The first derivative measures the rate of change
- The second derivative tells whether the change is "speeding up" or "speeding down"
- In other words, the second derivative indicates how the rate of change is itself changing

Higher-order derivatives

Demographers predict that t years from now the population of the world be :

$$P(t) = 6250 + 160t^{3/4}$$

million people. Find $P'(16)$ and $P''(16)$ and interpret.

Concave and convex functions

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Theorem

Suppose that f is continuous in the interval I and twice differentiable

- f is convex on $I \iff f''(x) \geq 0$ for all x in I
- f is concave on $I \iff f''(x) \leq 0$ for all x in I

Second-derivative test for minimum/maximum

Let f be a twice differentiable function in an interval I. Suppose c is an interior point of I. Then :

- $f'(c) = 0$ and $f''(c) < 0 \Longrightarrow c$ is a maximum point
- $f'(c) = 0$ and $f''(c) > 0 \Longrightarrow c$ is a minimum point
- $f'(c) = 0$ and $f''(c) = 0 \Longrightarrow$?