Exercise 1

1. \(f(x)\) is a well defined number for any \(x\).

2. The function is defined for all \(x\) such that \(2x + 4 \geq 0\), so for all \(x \in [-2, \infty]\).

3. The function is defined for all \(x\) such that \(x^2 - 4 \neq 0\), so for all \(x \neq 2\) and \(x \neq -2\).

4. The function is defined if \(i) \frac{x+1}{2-x} \geq 0\) and \(ii) 2 - x \neq 0\).
   \[x + 1 \leq 0 \text{ if and only if } x \leq -1.\]
   \[2 - x \leq 0 \text{ if and only if } x \geq 2.\]
   Hence, \(f(x)\) is defined for \(x \in [-1; 2[\).

Exercise 2

- \(f(0) = (0 - (0 - a)^2) = -(-a)^2 = -a^2\).
- \(f(a) = a^2 - (a - a)^2 = a^2\).
- \(f(-a) = (-a)^2 - (-a - a)^2 = a^2 - (-2a)^2 = a^2 - 4a^2 = -3a^2\).
- \(f(2a) = (2a)^2 - (2a - a)^2 = 4a^2 - (a)^2 = 3a^2\).

Exercise 3

- \(2x + 5y = 0 \iff y = -\frac{2}{5}x\). So the slope is \(-\frac{2}{5}\) and the intercept is 0.
- \(6x - 7y = 14 \iff y = \frac{6}{7}x - 2\). So the slope is \(\frac{6}{7}\) and the intercept is -2.
- \(3y - 9 = 0 \iff y = 3\). So the slope is 0 and the intercept is 3.
(a)

(b)

(c)
Exercise 4

• \( a = \frac{4-2}{4-(-2)} = \frac{1}{3} \). We know that \( y = \frac{1}{3}x + b \). We can substitute \( x \) and \( y \) in this equation by, say, \((4,4)\). Hence, \( 4 = \frac{1}{3}x + b \), so \( b = \frac{8}{3} \).

• \( y = ax+b = 2x+b \). We know that it goes through \((4,2)\), so \( 2 = 2 \times 4 + b \), hence \( b = -6 \).

Exercise 5

1. As the price of one unit of \( x \) is 3 and the price of one unit of \( y \) is 5, the budget constraint is \( 3x + 5y = 120 \). We can rewrite it as \( y = 24 - \frac{3}{5}x \).

All the bundles \((x,y)\) that cost 120 are on this line.

2. If the budget is cut by 25\%, we are left with 90 euros. Our new budget constraint is then \( 3x + 5y = 90 \).
Note that the budget line is parallel to the previous one: only the intercept changed.

3. If the price of $x$ doubles, one unit of $x$ now costs 6 euros. The budget constraint is then: $6x + 5y = 120$.

This time, this is the slope that changed: the relative prices changed!

Exercise 6

1. \[ \begin{align*} 6x + 3y &= 1 \\ 2x - 5y &= 5 \end{align*} \] 
   For this first example, we can use the elimination method.
   First, multiply equation (2) by 3: $6x - 15y = 15$. Now, you have $6x$ in the two equations.
   You can then compute (1) - (2) in order to get rid of the $x$: $0x + 18y = -14 \Rightarrow y = -\frac{7}{9}$. Now that you have a value for $y$, you simply have to insert it either in (1) or in (2) to get the value of $x$: $2x - 5 \times -\frac{7}{9} = 5 \Rightarrow x = \frac{5}{9}$.

2. \[ \begin{align*} x - y &= 8 \\ 2x + 3y &= 11 \end{align*} \] 
   For this example, it is easy to use the substitution method.
   First, rewrite (1): $x = 8 + y$. We can now substitute $x$ in (2) by $8 + y$:
   $2 \times (8 + y) + 5y = 11 \Rightarrow y = -1$. Now you can plug this value either in (1) or (2):
   $x + 1 = 8 \Rightarrow x = 7$.

3. \[ \begin{align*} \frac{5}{2}x - \frac{1}{2}y + \frac{1}{4} &= 0 \\ -x - \frac{7 - 3y}{2} &= 0 \end{align*} \]
You should obtain $x = 1$ and $y = 3$.

**Exercise 7**

For this exercise, you have to remember that $(a + b)^2 = a^2 + 2ab + b^2$.

We are looking for the values of $x$ satisfying $ax^2 + bx + c = 0$.

Let’s divide by $a$: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \iff x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Consider $x^2 + \frac{b}{a}x$. What do we need to add in order to get an expression similar to $a^2 + 2ab + b^2$?

To see clearer, we can rewrite it as: $x^2 + 2 \times \frac{b}{2a}x$.

So if we add to this expression $(\frac{b}{2a})^2$, we get $x^2 + \frac{b}{a}x + \frac{b}{2a}^2 = (x + \frac{b}{2a})^2$.

As we added $(\frac{b}{2a})^2$ on one side, we have to add it on the other side as well:

$\Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$.

Exercise 8

We are looking for the value of $x$ which minimizes $f(x) = ax^2 + bx + c$, knowing that $a$, $b$ and $c > 0$.

First, using the same trick as above, you can rewrite $f(x)$ as follows:

$f(x) = a(x^2 + \frac{b}{a}x) + c = a(x^2 + 2 \times \frac{b}{2a}x + \frac{b}{4a^2}) - a \times \frac{b^2}{4a^2} + c.$

$f(x) = a(x + \frac{b}{2a})^2 - a \times \frac{b^2}{4a^2} + c$.

$f(x) = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}$.

Note that the second term, $\frac{b^2 - 4ac}{4a}$, is a constant: it does not depend on $x$. So the minimum of $f(x)$ is reached when $(x + \frac{b}{2a})^2$ is minimized (remember that $a \geq 0$). As it is a squared term, it will never be negative. Thus, it reaches its minimum value when $(x + \frac{b}{2a})^2 = 0$, which implies that $x = -\frac{b}{2a}$.

Exercise 9

1. $f(x) = x^2 + 4x = 0 \iff x(x + 4) = 0$. So $x = 0$ or $x = -4$.

The minimum is reached for $x = -2$. 5
2. $f(x) = x^2 + 6x + 18 = 0$. The discriminant $b^2 - 4ac$ is negative: there is no real root. The minimum is reached for $x = -3$.

3. $f(x) = -3x^2 + 30x - 30 = 0 \quad \frac{-30 + \sqrt{1260}}{6} \approx 0.91; -30 - \sqrt{12606} \approx -10.91$. The maximum is reached for $x = -5$.

4. $f(x) = x^3 - 4x^2 - x = 0$. Take care! The quadratic formula is only valid for quadratic functions! To obtain the roots, we can factorize: $x^3 - 4x^2 - x = x(x^2 - 4x - 1) = 0$. So $x = 0$ or $x^2 - 4x - 1 = 0$, which is the case for $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$. If you plot the graph of this function, you will see that there is no global maximum nor global minimum.